Higher symmetries of the elliptic Euler-Darboux equation

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Abstract

We find a remarkable subalgebra of higher symmetries of the elliptic Euler-Darboux equation. To this aim we map such equation into its hyperbolic analogue already studied by Shemarulin. Taking into consideration how symmetries and recursion operators transform by this complex contact transformation, we explicitly give the structure of this Lie algebra and prove that it is finitely generated. Furthermore, higher symmetries depending on jets up to second order are explicitly computed.

Keywords: jet spaces, Euler-Darboux equation, complex transformation, higher symmetries, recursion operators.

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Introduction

In this paper we find a remarkable subalgebra of higher symmetries (also known as generalized symmetries, [8]) of the following linear elliptic equation:

$$\mathcal{E}_{ED} = \{(x+y)(u_{xx}+u_{yy}) + u_x + u_y = 0\} \tag{1}$$

in the unknown function $u = u(x,y)$. This Euler-Darboux type equation appears in two recent papers ([13], [14]) devoted to the study of Lorentzian Ricci flat 4-metrics admitting a bidimensional nonabelian Lie algebra $\mathcal{G}$ of Killing vector fields with non null orbits. Coordinates (x,y) appearing in (1) are a conformal chart on the Riemannian surface M, quotient of the space-time with respect to Killing foliation (see below for further details). Hence, (1) must be considered as (a local representation of) a second order equation on the trivial bundle $M \times \mathbb{R} \to M$, $(x,y,u) \to (x,y)$.

Let us briefly recall the physical origin of (1). It has been shown in [13] that, if the distribution orthogonal to the orbits is completely integrable and $\mathcal{G}$ admits a null Killing vector field Y, then the most general Ricci flat metric of this type takes, in a suitable chart (x_1,x_2,p,q), the form

$$g = 2f\left(dx_1^2 + dx_2^2\right) + \mu\left[\left(h(x_1,x_2) - 2q\right)dp^2 + 2dpdq\right], \tag{2}$$

where: $\mu = A\,\Phi + B$ with $A, B \in \mathbb{R}$, $\Phi(x_1, x_2)$ is a non constant harmonic function, $f = \pm\,(\nabla\Phi)^2 / \sqrt{|\mu|}$ and $h\,(x_1, x_2)$ is a solution of equation

$$\Delta h + (\partial_{x_1} \ln|\mu|)\,\partial_{x_1} h + (\partial_{x_2} \ln|\mu|)\,\partial_{x_2} h = 0. \tag{3}$$

Space-time metrics (2) were already known since the earlier works of Dautcourt, Ehlers, Kramer and Kundt (see [3], [4], [5], [7] and also Chapter 24 of [15]) on Ricci flat 4-metrics with a null Killing vector field.

In fact, one can distinguish two cases according to whether μ is constant or not. When μ is constant the above metric is a particular case of pp-wave (see [15]). On the contrary, if μ is not constant, by taking μ and its harmonic conjugate $\tilde{\mu}$ as new coordinates one can bring (3) to the more simple form

$$\mu\,(h_{\mu\mu} + h_{\tilde{\mu}\tilde{\mu}}\,) + h_{\mu} = 0 \tag{4}$$

and (2) takes the well known form (see [15])

$$g = \frac{1}{\sqrt{|\mu|}}\left(d\mu^2 + d\tilde{\mu}^2\right) + 2\mu dp\,(dq + Mdp) \tag{5}$$

with $M = h\,(\mu, \tilde{\mu})\,/2 - q$.

Hence, in order to find concrete Ricci flat metrics of the form (5) it is necessary to find exact solutions of (4) or, equivalently, equation (1) which is obtained by (4) through coordinate transformation $\{x = \mu + \tilde{\mu},\ y = \mu - \tilde{\mu}\}$.

The most efficient way to do this consists in finding classical and higher symmetries of such equations and using them to generate solutions (see next section for a brief recall of the notion of symmetry of a system of PDE's; for further details look at references therein).

Classical symmetries of (1) have been already studied in our previous paper [2]. The aim of the present paper is to describe a subalgebra of higher symmetries of (1) by reducing it, via a complex contact transformation H (see section 2), to its hyperbolic analogue

$$\mathcal{Y}_{ED} = \{2\,(\xi + \eta)\,u_{\xi\eta} + u_{\xi} + u_{\eta} = 0\}\,, \tag{6}$$

whose higher symmetries have been studied in detail in [10], [11] and [12]. Of course, using a complex transformation involves several problems of both geometrical and analytical kind. In particular, one must complexify the original bundle $\pi : M \times \mathbb{R} \to M$ and the corresponding jet bundles. Now, the complexification of a real analytical manifold can be done in several ways, all locally equivalent ([18]). However, we are interested in preserving the jet bundle structure and, furthermore, in the possibility of holomorphically extending real analytical functions to the *whole* complexified jet bundles. For these reasons, in the rest of the paper it will be assumed that: 1) $M = \mathbb{R}^2$; 2) the only functions and differential operators to be considered on the original real jet bundles are those which rationally depend on base and jet variables. In fact, as we are mainly interested in local coordinate expressions of infinitesimal symmetries of (1), the first assumption does not represent too severe a restriction. As to the rationality assumption, it has been made to ensure the holomorphic *global* extendability. In particular, we

construct a vector space isomorphism between rational higher symmetries of equations (6) and (1).

The paper is structured as follows.

In section 1 we give basic notions of the theory of symmetries of differential equations, which we interpret as submanifolds of jet bundles. Namely, we regard equation (1) as a submanifold of the second jet bundle $J^2(\pi)$. In this geometrical setting, higher symmetries are vector fields on the infinite prolongation of $\mathcal{E}_{ED}$ preserving its contact structure; in fact, such vector fields can be identified with the corresponding generating functions, so that we use the latter in concrete computations.

In section 2 we analyze the action of complex transformation H on a generic linear partial differential equation $\mathcal{E}$ in two independent variables, with rational coefficients. After having explicitly determined prolongation H and its most remarkable properties, we construct, starting from it, a vector space isomorphism Θ between the algebrae of rational higher symmetries of equation $\mathcal{E}$ and of transformed equation $\mathcal{Y}$; roughly speaking, Θ maps any rational symmetry φ of $\mathcal{Y}$ into the sum of the real and imaginary part of the pullback of φ along H (theorem 6). By a completely analogous reasoning one obtains a corresponding isomorphism between the rational recursion operators of the two equations (theorem 10).

In section 3 we give concrete examples of computation, taking into consideration the results of [10], [11] and [12]. We explicitly find symmetries of $\mathcal{E}_{ED}$ depending on second derivatives, obtaining as a by-product contact symmetries already found in [2]. Finally, up to solutions of $\mathcal{E}_{ED}$, the Lie algebra structure of rational higher symmetries of $\mathcal{E}_{ED}$ is completely determined.

1 Basic notions on symmetries

In this section we recall the basics about jet bundles and symmetries of PDE (for further details see [1],[6],[8],[9],[16],[17]).

Let M be an n-dimensional smooth manifold and $\pi\colon E \to M$ be a vector bundle, $\dim E = n+m$. Let $\mathcal{U} \subset M$ be a neighborhood of M such that $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{R}^m$ and let (x_λ, u^i), $\lambda = 1 \dots n$, $i = 1 \dots m$, with (x_λ) coordinates on $\mathcal{U}$, be the corresponding trivialization. Then a local section of π is locally given by $u^i = f^i(x_1, x_2, \dots, x_n)$. We shall denote by $\Gamma(\pi)$ the $C^\infty(M)$-module of local sections of π.

Two local sections s and $\tilde{s}$ of π are said to be *r-contact equivalent* at the point $x \in M$ if their Taylor expansions at this point coincide up to order r. This is an equivalence relation, and we shall denote by $[s]_x^r$ an equivalence class. The set $J^r(\pi)$ of all the equivalence classes $[s]_x^r$ is called the *jet bundle* of order r and it has a natural vector bundle structure. A chart (x_λ, u^i_σ) on $J^r(\pi)$ is defined by $u^i_\sigma([s]_x^r) = \frac{\partial^{|\sigma|} f^i}{\partial x_\sigma}(x)$, where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ with $|\sigma| \overset{\text{def}}{=} \sum \sigma_i \leq r$ and $0 \leq \sigma_i \leq n$, is a multi-index and $\frac{\partial^{|\sigma|}}{\partial x_\sigma}$ stands for $\frac{\partial^{|\sigma|}}{\partial x_1^{\sigma_1} \cdots \partial x_n^{\sigma_n}}$.

We have the following natural maps:

1. the embedding $j_r s\colon M \to J^r(\pi),\ x \mapsto [s]^r_x$,
2. the projections $\pi_{k,h}\colon J^k(\pi) \to J^h(\pi),\ [s]^k_x \mapsto [s]^h_x \quad k \geq h$,
3. The base projections $\pi_r : J^r(\pi) \to M,\ \ [s]^r_x \mapsto x$.

Note that there is a natural bijection between smooth functions on $J^r(\pi)$ and scalar r-th order differential operators on $\Gamma(\pi)$. Namely, with each $F \in C^\infty(J^r(\pi))$ one can associate the following operator

$$\Delta_F : \Gamma(\pi) \to C^\infty(M)\,,\ \ s \mapsto F \circ j_r s, \quad s \in \Gamma(\pi)\,.$$

The *contact plane* $\mathcal{C}^r_\theta$ at the point $\theta \in J^r(\pi)$ is the span of the planes $T_\theta(j_r s(M))$, with $s \in \Gamma(\pi)$ varying among sections whose r-jet at $\pi_r(\theta)$ coincides with θ. We have the *contact distribution* $\theta \mapsto \mathcal{C}^r_\theta$ on $J^r(\pi)$. A diffeomorphism of $J^r(\pi)$ is called a *contact transformation* if it is a symmetry of the contact distribution (i.e. if it preserves contact planes). A vector field on $J^r(\pi)$ whose flow consists of contact transformations is called a *contact field*. We note that a point $\theta = [s]^{r+1}_x$ of $J^{r+1}(\pi)$ is completely characterized by $T_{\pi_{r+1,r}(\theta)}(j_r s(M))$. Then we can lift a contact transformation G of $J^r(\pi)$ to a contact transformation $G^{(1)}$ of $J^{r+1}(\pi)$ by considering $G_*(T_{\pi_{r+1,r}(\theta)}(j_r s(M)))$. Of course we can lift contact fields by lifting their local flows. According to a classical result by Lie and Baecklund, any contact transformation is the lifting: 1) of a first order contact transformation if $m = \operatorname{rank}\pi = 1$; 2) of a diffeomorphism of $J^0(\pi) = E$ if $m > 1$. An analogous result holds for contact fields.

A *differential equation* $\mathcal{E}$ of order r is a submanifold of $J^r(\pi)$. A *linear* equation is a linear subbundle of $J^r(\pi) \to M$. A (local) *solution* of $\mathcal{E}$ is a section s of π such that $j_r s(M) \subset \mathcal{E}$. The *1-prolongation* $\mathcal{E}^1$ of the equation $\mathcal{E}$ is the set of first order "differential consequences" of $\mathcal{E}$. Geometrically:

$$\mathcal{E}^1 = \{[s]^{r+1}_x \in J^{r+1}(\pi)\,|\, s \in \Gamma(\pi),\ [s]^r_x \in \mathcal{E},\ T_{[s]^r_x}(j_r s(M)) \subset T_{[s]^r_x}\mathcal{E}\}.$$

By iteration we can define the l-prolongation $\mathcal{E}^l$. Locally, if the equation $\mathcal{E}$ is described by $\{F^i = 0\}$, with $F^i \in C^\infty(J^r(\pi))$, then $\mathcal{E}^l$ is described by $\{D_\sigma(F^i) = 0\}$ with $0 \leq |\sigma| \leq l$, where $D_\sigma = D_{\sigma_1} \circ D_{\sigma_2} \circ \cdots \circ D_{\sigma_n}$ and D_λ are *total derivatives*:

$$D_\lambda = \frac{\partial}{\partial x_\lambda} + \sum_{j,\sigma} u^j_{\sigma,\lambda} \frac{\partial}{\partial u^j_\sigma}.$$

Note that the above definitions of r-contact equivalence and r-th order jet space make sense even in the case $r = \infty$. Obviously, $J^\infty(\pi)$ is *not* a finite dimensional smooth manifold (its points are sequences of the form $\{\theta_r\}$, $r \in \mathbb{N}_0$, with $\theta_r \in J^r(\pi)$ and $\pi_{r,r-1}(\theta_r) = \theta_{r-1}$). However, a very rich differential calculus can be developed on it, making it an extremely useful tool in symmetry analysis of PDE's as well as in many other fields. Here we limit ourselves to recall just a few basic facts about the differential structure on $J^\infty(\pi)$ (for further details see [1]):

By definition, smooth functions on $J^\infty(\pi)$ are pullbacks of smooth functions on finite order jet spaces along projections $\pi_{\infty,k}$. Thus, $C^\infty(J^\infty(\pi))$ is a filtered algebra (the degree being the jet order of the pullbacked function). Consequently, vector fields on $J^\infty(\pi)$ are defined as derivations $X : C^\infty(J^\infty(\pi)) \to C^\infty(J^\infty(\pi))$ such that $\deg X(f) - \deg f$ is a constant integer depending only on X. Vector fields on $J^\infty(\pi)$ do not admit, generally, a flow, even locally. For instance, D_λ is a vector field on $J^\infty(\pi)$ with degree 1. A tangent vector at a point $\theta = \{\theta_r\} \in J^\infty(\pi)$ is a sequence $\xi = \{\xi_r\}$ such that $\xi_r \in T_{\theta_r} J^r(\pi)$ and $d_{\theta_r}\pi_{r,r-1}(\xi_r) = \xi_{r-1}$. The contact plane $\mathcal{C}_\theta$ is the sequence $\{\mathcal{C}^r_{\theta_r}\}$. Contact distribution $\theta \longmapsto \mathcal{C}_\theta$ on $J^\infty(\pi)$ is n-dimensional (it is spanned by total derivatives $\{D_\lambda\}_{1\le\lambda\le n}$) and formally integrable, in the sense that its generators satisfy Frobenius conditions. A vector field on $J^\infty(\pi)$ lying in the contact distribution $\mathcal{C}$ is called *trivial* as it is tangent to all integral manifolds of $\mathcal{C}$. Any contact field X on $J^\infty(\pi)$ can be splitted in a vertical and a trivial part. More precisely we have that:

$$X = X_\varphi + T$$

where

$$X_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j)\frac{\partial}{\partial u^j_\sigma}, \tag{7}$$

with $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_j \in C^\infty(J^\infty(\pi))$ and T is a trivial vector field. It can be proved that fields of the form (7), called *evolutionary vector fields*, are the only vertical contact fields on $J^\infty(\pi)$; vector function φ is called the *generating section* of X_φ (also know as characteristic of a contact field, [8]). Evolutionary vector fields form a Lie subalgebra, isomorphic to the algebra of generating sections with respect to *Jacobi bracket*

$$\{\varphi, \psi\} \overset{\text{def}}{=} X_\varphi(\psi) - X_\psi(\varphi) \tag{8}$$

A *classical symmetry* of $\mathcal{E}$ is a contact field on $J^r(\pi)$ tangent to $\mathcal{E}$. If, in particular, it is a lift of a vector field on E, then it is called a *point symmetry.* A contact field on $J^\infty(\pi)$ tangent to $\mathcal{E}^\infty$ is called an *external higher symmetry.* A vector field on $\mathcal{E}^\infty$ which preserves the contact distribution induced on $\mathcal{E}^\infty$ is called an *internal higher symmetry.*

Now we are interested in non-trivial symmetries, that is symmetries of the form X_φ. Locally, if the equation $\mathcal{E}$ is described by $\{F^i = 0\}$, with $F^i \in C^\infty(J^r(\pi))$, then the vector field X_φ is a symmetry of $\mathcal{E}$ iff $X_\varphi(F^i)|_{\mathcal{E}^\infty} = 0$. If we define matrix operator ℓ_F by

$$\ell_F(\varphi) \overset{\text{def}}{=} \left\| X_{\varphi^j}\left(F^i\right)\right\|$$

we have that φ is an external higher symmetry of $\mathcal{E}$ if and only if

$$(\ell_F(\varphi))\,|_{\mathcal{E}^\infty} = 0. \tag{9}$$

The operator $\ell_\mathcal{E} = \ell_F|_{\mathcal{E}^\infty}$ is called *universal linearization* of $\mathcal{E}$. Locally we have that φ is an external higher symmetry if

$$\ell_\mathcal{E}(\bar\varphi) = \sum_{j,\sigma} \frac{\partial F^i}{\partial u^j_\sigma}\bar D_\sigma(\bar\varphi^j) = 0 \tag{10}$$

where the bar denotes the restriction to $\mathcal{E}^\infty$.

It is easy to realize that any external higher symmetry restricts to an internal higher symmetry. The converse is also true: each internal higher symmetry can be obtained by restricting on $\mathcal{E}^\infty$ some external one. For this reason we shall not distinguish them, and we shall call them simply higher symmetries. Then we shall denote by $\mathrm{Sym}(\mathcal{E})$ the algebra of (non-trivial) higher symmetries of $\mathcal{E}$.

A vector valued operator Δ acting on vector functions on $J^\infty(\pi)$ is called $\mathcal{C}$*-differential* if its restriction to $\mathcal{E}^\infty$ is well defined for any differential equation $\mathcal{E}$. In local coordinates, Δ reads

$$\Delta = \left\| \sum_\sigma a_{ij}^\sigma D_\sigma \right\|, \quad \text{where} \quad a_{ij}^\sigma \in C^\infty(J^\infty(\pi)).$$

An example of $\mathcal{C}$-differential operator is given by ℓ_F.

Finally, a *recursion operator* $\Re \in \mathrm{Rec}(\mathcal{E})$ for a differential equation $\mathcal{E}$ is a linear $\mathcal{C}$-differential operator which maps $\mathrm{Sym}(\mathcal{E})$ into itself.

2 Symmetries and recursion operators of linear equations by complex transformations

In this section it will be shown how do symmetries of a scalar linear PDE in two independent variables transform under a complex point transformation. As we are mainly interested in local coordinate expressions of infinitesimal symmetries, we can safely assume that the bundle of independent and dependent variables is the trivial bundle $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$.

Consider the r-th order scalar PDE

$$\mathcal{E} = \{F\left(x, y, u, u_x, u_y, ..., u_{hx,ky}, ..., u_{ry}\right) = 0\} \tag{11}$$

where (x, y, u) are standard coordinates on $\mathbb{R}^2 \times \mathbb{R}$, $u_{hx,ky} \overset{\text{def}}{=} u_{\underbrace{xx...x}_{\text{h-times}}\underbrace{yy...y}_{\text{k-times}}}$, $h+k \leq r$, and F is assumed to be linear in $u, u_x, u_y, ..., u_{hx,ky}, ..., u_{ry}$ with coefficients which are rational functions in x, y (the reason of this assumption has been given in the introduction). We want to express $\mathcal{E}$ in terms of complex conjugate variables $z, \overline{z}$, with $z = x + iy$. The complex analogues of jet variables are defined as follows. Recall that to $z, \overline{z}$ one can associate formal partial derivatives:

$$\frac{\partial}{\partial z} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{12}$$

These are vector fields on the complexified tangent bundle $T_{\mathbb{C}}\left(\mathbb{R}^2\right) \simeq \mathbb{R}^2 \times \mathbb{C}^2$. Let also $\pi_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$ be the complexified bundle of π. One can immediately extend to it the jet bundle construction of section 1. Let $s = s_1 + is_2$ be a section of $\pi_{\mathbb{C}}$ and $\theta = [s]_a^k$ be its k-jet at point $a \in \mathbb{R}^2$. Then its complex conjugate is, by definition, $\overline{\theta} = [\overline{s}]_a^k$ and its jet coordinates $\overline{u}_{rx,py}$ are complex conjugate to those $u_{rx,py}$ of θ.

Starting from (12) one defines by iterated composition higher order partial derivatives of sections of $\pi_{\mathbb{C}}$ with respect to base "coordinates" z, $\overline{z}$. This allows to define complex jet coordinates of θ as

$$u_{hz,l\overline{z}} \overset{\mathrm{def}}{=} \frac{\partial^{h+l} s}{\partial z^h \partial \overline{z}^l}(a) \,, \tag{13}$$

for $0 \leq h+l \leq k$.

Proposition 1 *Let $s \in \Gamma(\pi_{\mathbb{C}})$. Then*

$$\overline{\left(\frac{\partial^{h+l} s}{\partial z^h \partial \overline{z}^l}\right)} = \frac{\partial^{h+l} \overline{s}}{\partial z^k \partial \overline{z}^l}, \tag{14}$$

for any $h, l \in \mathbb{N}$.

Proof. It easily follows from (12) by induction . ■

An immediate consequence of the previous proposition is the following relation between k-th order complex jet variables of *real* jets ($\theta = \overline{\theta}$):

$$\overline{u_{(k-r)z,r\overline{z}}} = u_{rz,(k-r)\overline{z}}, \tag{15}$$

Analogously, one can define also total derivatives with respect to z, $\overline{z}$. First order total derivatives are, by definition

$$D_z = \frac{1}{2}\left(D_x - iD_y\right), \qquad D_{\overline{z}} = \frac{1}{2}\left(D_x + iD_y\right) \tag{16}$$

and higher order total derivatives are obtained by (16) via iterated composition.

Complex jet variables are related to real ones by linear relation:

$$\left(z, \overline{z}, u, u_z, u_{\overline{z}}, \ldots\right)^T = H \cdot \left(x, y, u, u_x, u_y, \ldots\right)^T, \tag{17}$$

where H is the (infinite) block matrix

$$H = \begin{pmatrix} P & & & & & & \\ & 1 & & & & & \\ & & P^{(1)} & & & & \\ & & & P^{(2)} & & & \\ & & & & \ddots & & \\ & & & & & P^{(k)} & \\ & & & & & & \ddots \end{pmatrix},$$

with

$$P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{18}$$

and $P^{(k)}$ defined by

$$V^{(k)} = P^{(k)} U^{(k)}, \tag{19}$$

with $V^{(k)} = \left(u_{kz}, u_{(k-1)z,\overline{z}}, ..., u_{k\overline{z}}\right)^T$, $U^{(k)} = \left(u_{kx}, u_{(k-1)x,y}, ..., u_{ky}\right)^T$, $P^{(k)} = \left\|p^k_{rs}\right\|_{r,s=0,...,k}$. Any block is obtained by the previous one via recursive formulae (29) below. Therefore, the whole matrix H is determined by the first block P: in fact, H is just the contact prolongation of coordinate transformation H_0 on $J^0(\pi_{\mathbb{C}})$, given by:

$$(z, \overline{z})^T = P \cdot (x, y)^T, u = u. \tag{20}$$

In next section we will explicitly determine coefficients of matrix H and show some remarkable properties of it.

2.1 Explicit formulae for matrix H

It is clear that, for any $k \in \mathbb{N}$ and $r = 0, 1, \cdots, k$, $D_{(k-r)z,r\overline{z}}$ is a linear combination of standard total derivatives of order k:

$$D_{(k-r)z,r\overline{z}} = \sum_{q=0}^{r} p^k_{rq} D_{(k-q)x,qy}. \tag{21}$$

Consequently, keeping in mind that, as in the real case, complex jet variables can be expressed by formula

$$u_{hz,k\overline{z}} = D_{hz,k\overline{z}}(u), \tag{22}$$

with $D_{hz,k\overline{z}} = D^h_z \circ D^k_{\overline{z}}$, we have that

$$u_{(k-r)z,r\overline{z}} = \sum_{q=0}^{k} p^k_{rq}\, u_{(k-q)x,qy} \tag{23}$$

In order to compute coefficient p^k_{rs}, let us apply both hands of (21) to the function $x^{k-s}y^s$. Such a function depends only on base variables, so its total derivatives coincide with the corresponding partial ones. Hence, the only non zero term in the right hand side of (21) applied to $x^{k-s}y^s$ is that with $q = s$, i.e.

$$D_{(k-r)z,r\overline{z}}\left(x^{k-s}y^s\right) = s!\,(k-s)!p^k_{rs}$$

Hence

$$p^k_{rs} = \frac{1}{s!\,(k-s)!} D_{(k-r)z,r\overline{z}}\left(x^{k-s}y^s\right) \tag{24}$$

To compute the total derivative in (24) it is convenient to express $x^{k-s}y^s$ in terms of z and $\overline{z}$:

$$\begin{aligned} x^{k-s}y^s &= \frac{i^s}{2^k}(z+\overline{z})^{k-s}(\overline{z}-z)^s \\ &= \frac{i^s}{2^k}\sum_{\alpha=0}^{k-s}\sum_{\beta=0}^{s}(-1)^\beta \binom{k-s}{\alpha}\binom{s}{\beta} z^{\alpha+\beta}\overline{z}^{k-\alpha-\beta} \end{aligned}$$

Now, following the same argument as above, the only terms in this double sum on which $D_{(k-r)z,r\overline{z}}$ does not vanish are those corresponding to the monomial $z^{k-r}\overline{z}^r$, i.e.

$$\begin{aligned}
&\frac{i^s}{2^k}\sum_{\alpha+\beta=k-r}(-1)^\beta\binom{k-s}{\alpha}\binom{s}{\beta}z^{\alpha+\beta}\overline{z}^{k-\alpha-\beta}\\
&=(-1)^{k-r}\frac{i^s}{2^k}\sum_{\alpha=M(k,r,s)}^{m(k,r,s)}(-1)^\alpha\binom{k-s}{\alpha}\binom{s}{k-r-\alpha}z^{k-r}\overline{z}^r,
\end{aligned}$$

with $M(k,r,s)=\max(0,k-r-s)$, $m(k,r,s)=\min(k-r,k-s)$. Hence, (24) can be rewritten as follows:

$$\begin{aligned}
p_{rs}^k&=(-1)^{k-r}\frac{r!\,(k-r)!}{2^k s!\,(k-s)!}i^s\sum_{\alpha=M(k,r,s)}^{m(k,r,s)}(-1)^\alpha\binom{k-s}{\alpha}\binom{s}{k-r-\alpha} \qquad (25)\\
&=(-1)^{k-r}\frac{r!\,(k-r)!}{2^k}i^s\sum_{\alpha=M(k,r,s)}^{m(k,r,s)}\frac{(-1)^\alpha}{\alpha!\,(k-s-\alpha)!\,(k-r-\alpha)!\,(r+s+\alpha-k)!}
\end{aligned}$$

2.1.1 Some remarkable properties of matrix H

- It is obvious that, for any $k\in\mathbf{N}$, the block $P^{(k)}$ is invertible (one must be able to express k-th order standard jet variables in term of complex ones). It can be proved that the inverse is ${P^{(k)}}^{-1}=\left\|q_{rs}^k\right\|$, with

$$q_{rs}^k=2^k i^{r+s}p_{rs}^k$$

- For each block $P^{(k)}$ the following symmetry property holds

$$\overline{p_{rq}^k}=p_{k-r,q}^k, \qquad (26)$$

  for $r,q=0,\ldots,k$. This is an immediate consequence of (23) and (15).

- For any k, $\overline{P^{(k)}}\cdot{P^{(k)}}^{-1}$ is real. In fact, considering, as above, only real sections of $\pi_{\mathbb{C}}$ and taking the complex conjugate of both hands of (19), one gets

$$\overline{V^{(k)}}=\overline{P^{(k)}}U^{(k)}=\overline{P^{(k)}}{P^{(k)}}^{-1}V^{(k)} \qquad (27)$$

  But, keeping in mind (15), one also gets

$$\overline{V^{(k)}}=AV^{(k)}, \qquad (28)$$

  where $A=\left\|\delta_{i,k-j}\right\|_{i,j=0,\ldots,k}$. This, together with (27), implies that $\overline{P^{(k)}}{P^{(k)}}^{-1}=A$

- The following recurrence formulae hold:

$$
\begin{aligned}
p_{r,s}^{k+1} &= \frac{1}{2}\left(p_{r,s}^{k} - i p_{r,s-1}^{k}\right) \qquad \text{for } r \leq k \\
p_{k+1,s}^{k+1} &= \frac{1}{2}\left(p_{k,s}^{k} + i p_{k,s-1}^{k}\right),
\end{aligned}
\tag{29}
$$

where we pose $p_{r,-1}^{k} = p_{r,k+1}^{k} = 0$ for $r = 0, 1, \cdots k$. Such formulae are obtained from relations $u_{(k+1-r)z,r\overline{z}} = D_z\left(u_{(k-r)z,r\overline{z}}\right)$, $u_{(k+1)\overline{z}} = D_{\overline{z}}\left(u_{k\overline{z}}\right)$ together with (16) e (23).

2.2 The transformed equation and its symmetries

Before proving the main result of this section, i.e. theorem 6, some remarks about linear transformation (17) are necessary.

First of all, (17) can be extended to complex values of variables $(x, y, u, u_x, u_y, \ldots)$; in other words, H can be viewed as a contact transformation

$$
(\xi, \eta, u, u_\xi, u_\eta, \ldots)^T = H \cdot (x, y, u, u_x, u_y, \ldots)^T \tag{30}
$$

of a complex jet space defined as follows.

Recall that a complexification of a real analytic manifold N is a complex manifold $\widehat{N}$ together with a real analytical embedding $\alpha : N \to \widehat{N}$ and an involutive antiholomorphism $\chi : \widehat{N} \to \widehat{N}$ such that $\chi \circ \alpha = \alpha$. As in the previous section, let π be the trivial bundle $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. Below we construct a complexification of $J^\infty(\pi)$ which is the right domain for transformation (30).

Let $\widehat{\pi}$ be the trivial holomorphic bundle $\mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}^2$. $\widehat{\pi}$ is a natural complexification of π and transformation (20) can be extended to the following automorphism of $\widehat{\pi}$:

$$
(\xi, \eta)^T = P \cdot (x, y)^T, \quad u = u,
$$

with $(x, y, u) \in \mathbb{C}^2 \times \mathbb{C}$. It is clear that the jet bundle construction described in section 1 can be repeated word by word for holomorphic jet bundles. Thus, starting from $\widehat{\pi}$ one gets a sequence of jet bundles $J^k(\widehat{\pi})$ whose inverse limit we denote by $J^\infty(\widehat{\pi})$. It is easily seen that, for any $k \in \mathbb{N}$, $J^k(\widehat{\pi})$ is a complexification of $J^k(\pi)$. In fact, let $\iota_k : J^k(\pi) \to J^k(\widehat{\pi})$ be defined in the following way. For any $s \in \Gamma(\pi)$, $a \in \mathbb{R}^2$, denote by $P_{s,a}^k : \mathbb{R}^2 \to \mathbb{R}$ the k-th order Taylor polynomial of s at a. Then

$$
\iota_k\left([s]_a^k\right) \stackrel{\text{def}}{=} \left[\widehat{P}_{s,a}^k\right]_a^k,
$$

where $\widehat{P}_{s,a}^k : \mathbb{C}^2 \to \mathbb{C}$ is the complex extension of $P_{s,a}^k$. Furthermore, let $\sigma_k : J^k(\widehat{\pi}) \to J^k(\widehat{\pi})$ be the map $[s]_a^k \mapsto [\overline{s}]_a^k$. The triple $\left(J^k(\widehat{\pi}), \iota_k, \sigma_k\right)$ is the required complexification of $J^k(\pi)$. The inverse limit $J^\infty(\widehat{\pi})$ is a complexification of $J^\infty(\pi)$ in the following sense. Let $\iota : J^\infty(\pi) \to J^\infty(\widehat{\pi})$ be the map defined by

$$
\iota(\theta) = \{\iota_k(\theta_k)\}_{k \in \mathbb{N}_0},
$$

with $\theta = \{\theta_k\}_{k\in\mathbb{N}_0}$. Analogously, the conjugate map $\sigma : J^\infty(\widehat{\pi}) \to J^\infty(\widehat{\pi})$ is defined as the inverse limit of $\{\sigma_k\}_{k\in\mathbb{N}_0}$. So, by the phrase "$J^\infty(\widehat{\pi})$ is a complexification of $J^\infty(\pi)$" we simply mean that we consider $J^\infty(\widehat{\pi})$ together with maps ι and σ. We stress that, in accordance with the real case (see section 1), one must consider as holomorphic functions on $J^\infty(\widehat{\pi})$ only the pullbacks of holomorphic functions on finite order jet bundles $J^k(\widehat{\pi})$.

We introduce the following definitions and notations. Denote by:

- $\Omega(J^\infty(\pi))$ the algebra of real analytic functions on $J^\infty(\pi)$, i.e. (see section 1) pullbacks of analytic functions on finite order jet bundles along projections $\pi_{\infty,k}$
- $\Omega(J^\infty(\widehat{\pi}))$ the algebra of complex analytic functions on $J^\infty(\widehat{\pi})$, in the same sense as the point above.
- $\Omega_{\mathbb{C}}(J^\infty(\pi)) \stackrel{\text{def}}{=} \{f_1 + if_2 \mid f_1, f_2 \in \Omega(J^\infty(\pi))\}$.

Under the action of H equation (11) is transformed into equation

$$\mathcal{Y} = \left\{\widetilde{F}(\xi, \eta, u, \cdots, u_{r\eta}) = 0\right\}, \tag{31}$$

with $\widetilde{F} = \widehat{F} \circ H^{-1} \circ \iota$. Here by $\widehat{F}$ we mean the holomorphic extension of F to $J^\infty(\widehat{\pi})$. Note that such an extension is well defined due to the particular form of F.

As we noted in the introduction, analytic functions on $J^\infty(\pi)$ are not generally holomorphically extendable on the complexification. Due to this reason, below we shall restrict our attention to rational functions of $x, y, u, u_x, u_y, ..., u_{hx,ky}, ...$ In other words, we consider, instead of $\Omega(J^\infty(\widehat{\pi}))$, the subalgebra $\mathcal{R}(J^\infty(\widehat{\pi}))$ of complex rational functions of (a finite number of) jet variables. Analogously, we denote by $\mathcal{R}(J^\infty(\pi)) \subset \Omega(J^\infty(\pi))$ the subalgebra of real rational functions on $J^\infty(\pi)$ and by $\mathcal{R}_{\mathbb{C}}(J^\infty(\pi)) \subset \Omega_{\mathbb{C}}(J^\infty(\pi))$ the algebra of functions of the form $f_1 + if_2$, with $f_i \in \mathcal{R}(J^\infty(\pi))$.

With these restrictions, the holomorphic extension operator

$$\rho : \mathcal{R}_{\mathbb{C}}(J^\infty(\pi)) \to \mathcal{R}(J^\infty(\widehat{\pi}))$$

is well defined (as above, we will occasionally denote $\rho(f)$ by $\widehat{f}$).

Below we will need to consider also holomorphic extension of differential operators.

Obviously, any linear differential operator on $\mathcal{R}(J^\infty(\pi))$ with rational coefficients can be extended by $\mathbb{C}$-linearity to an operator on $\mathcal{R}_{\mathbb{C}}(J^\infty(\pi))$.

With any differential operator $\Delta : \mathcal{R}_{\mathbb{C}}(J^\infty(\pi)) \to \mathcal{R}_{\mathbb{C}}(J^\infty(\pi))$ one associates its image $H(\Delta) : \mathcal{R}_{\mathbb{C}}(J^\infty(\pi)) \to \mathcal{R}_{\mathbb{C}}(J^\infty(\pi))$, defined as follows

$$H(\Delta) \stackrel{\text{def}}{=} \mathcal{H}^{-1} \circ \Delta \circ \mathcal{H}, \tag{32}$$

where

$$\mathcal{H} = \iota^* \circ H^* \circ \rho$$

Obviously, an analogous formula holds for the inverse map H^{-1}. Furthermore, with any differential operator $\Box : \mathcal{R}_{\mathbb{C}}\left(J^{\infty}\left(\pi\right)\right) \rightarrow \mathcal{R}_{\mathbb{C}}\left(J^{\infty}\left(\pi\right)\right)$ one associates its real part $\Box_1$ and its imaginary part $\Box_2$, define by

$$\Box_j\left(f\right) = \left(\Box\left(f\right)\right)_j ,$$

for any $f \in \mathcal{R}_{\mathbb{C}}\left(J^{\infty}\left(\pi\right)\right)$, $j = 1, 2$.

The linearization operators of equations (11) and (31) are related as follows:

$$\ell_{\widetilde{F}} = H\left(\ell_F\right) \tag{33}$$

Relation (33) is an immediate consequence of (10) and the identity $H\left(D_{rx,sy}\right) = \left(H\left(D_x\right)\right)^r \circ \left(H\left(D_y\right)\right)^s$.

Remark 2 *In our case, the general scheme above applies to equations (1) and (6). More exactly, (6) is the complex transformation of equation*

$$u_{XX} + u_{YY} + \frac{u_X}{X} = 0, \tag{34}$$

which in its turn is obtained by equation (1) by point transformation

$$G : (x, y, u) \rightarrow (X, Y, u)\,, \quad \textit{with} \quad X = x + y\ ,\ Y = x - y. \tag{35}$$

In the last part of this section we prove the main result of this paper. Namely, we construct (see theorem 6) a *vector space* isomorphism Θ from the algebra of real rational higher symmetries of $\mathcal{E}$ into that of $\mathcal{Y}$.

In order to prove theorem 6 some technical lemmata are necessary.

First of all, a straightforward computation shows that, for any vector field X on $J^{\infty}\left(\pi\right)$, the restrictions of $\left(H^{-1}(X)\right)_1$ and $\left(H^{-1}(X)\right)_2$ to $\mathcal{R}\left(J^{\infty}\left(\pi\right)\right)$ are vector fields on $J^{\infty}(\pi)$.

Lemma 3 *Let $\varphi \in \mathcal{R}\left(J^{\infty}\left(\pi\right)\right)$ be the generating function of the evolutionary vector field X_{φ}. Then we have that*

$$H^{-1}(X_{\varphi}) = X_{\mathcal{H}(\varphi)} \stackrel{def}{=} X_{(\mathcal{H}(\varphi))_1} + iX_{(\mathcal{H}(\varphi))_2} \tag{36}$$

Proof. Firstly $(H^{-1}(X_{\varphi}))_1$ and $(H^{-1}(X_{\varphi}))_2$ are vertical vector fields. This is an immediate consequence of verticality of X_{φ} and equalities: $\mathcal{H}^{-1}\left(x\right) = \frac{1}{2}\left(\xi + \eta\right)$, $\mathcal{H}^{-1}\left(y\right) = \frac{i}{2}\left(\eta - \xi\right)$.

Secondly, they are contact fields. In fact

$$\begin{aligned}
[(H^{-1}(X_{\varphi}))_j, D_x] = [H^{-1}(X_{\varphi}), D_x]_j &= \left(H^{-1}[X_{\varphi}, H(D_x)]\right)_j \\
&= \left(H^{-1}[X_{\varphi}, D_{\xi} + D_{\eta}]\right)_j \\
&= \left(H^{-1}[X_{\varphi}, D_{\xi}]\right)_j + \left(H^{-1}[X_{\varphi}, D_{\eta}]\right)_j
\end{aligned}$$

and analogously

$$[(H^{-1}(X_\varphi))_j, D_y] = \left(H^{-1}[X_\varphi, D_\xi]\right)_j - \left(H^{-1}[X_\varphi, D_\eta]\right)_j .$$

Finally $(\mathcal{H}(\varphi))_j$ is the corresponding generating section of $(H^{-1}(X_\varphi))_j$. In fact the generating section of $(H^{-1}(X_\varphi))_j$ is $(H^{-1}(X_\varphi))_j\,(u)$, and

$$\begin{aligned}(H^{-1}(X_\varphi))_j(u) &= \left(\mathcal{H}\circ X_\varphi \circ \mathcal{H}^{-1}\right)_j(u) = (\mathcal{H}(X_\varphi(\mathcal{H}^{-1}(u))))_j \\ &= (\mathcal{H}(X_\varphi(u)))_j = (\mathcal{H}(\varphi))_j.\end{aligned}$$

■

Denote by $\mathrm{Sym}_{\mathcal{R}}(\mathcal{Y})$ and $\mathrm{Sym}_{\mathcal{R}}(\mathcal{E})$, respectively, the algebra of rational symmetries of $\mathcal{Y}$ and $\mathcal{E}$.

Lemma 4 *If $\varphi \in \mathrm{Sym}_{\mathcal{R}}(\mathcal{Y})$ then $(\mathcal{H}(\varphi))_1$ and $(\mathcal{H}(\varphi))_2$ belong to $\mathrm{Sym}_{\mathcal{R}}(\mathcal{E})$.*

Proof. It is a direct consequence of (33). ■

Lemma 5 *Let $\varphi \in \mathcal{R}(J^\infty(\pi))$. Then*

1) $\left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_1 - \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_2 = \varphi$

2) $\left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_2 = \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_1 = 0$

Proof. We have that

$$\begin{aligned}\varphi &= \mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right) = \mathcal{H}^{-1}\left(\left(\mathcal{H}\left(\varphi\right)\right)_1 + i\left(\mathcal{H}\left(\varphi\right)\right)_2\right) = \mathcal{H}^{-1}\left(\left(\mathcal{H}\left(\varphi\right)\right)_1\right) + i\mathcal{H}^{-1}\left(\left(\mathcal{H}\left(\varphi\right)\right)_2\right) \\ &= \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_1 - \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_2 + i\left(\left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_2 + \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_1\right)\end{aligned}$$

As φ is a real function, relations 1) and

$$\left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_2 + \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_1 = 0 \tag{37}$$

hold. Also, $\mathcal{H}^{-1}\left(\overline{\mathcal{H}\left(\varphi\right)}\right)$ is real. In fact, first of all it holds:

$$\overline{\mathcal{H}\left(\varphi\right)} = \overline{\mathcal{H}}\left(\varphi\right), \tag{38}$$

where $\overline{\mathcal{H}} : \mathcal{R}_{\mathbb{C}}\left(J^\infty\left(\pi\right)\right) \to \mathcal{R}_{\mathbb{C}}\left(J^\infty\left(\pi\right)\right)$ is the map associated with the complex conjugate of matrix H. Then

$$\mathcal{H}^{-1}\left(\overline{\mathcal{H}\left(\varphi\right)}\right) = \mathcal{H}^{-1}\left(\overline{\mathcal{H}}\left(\varphi\right)\right) = \left(\overline{H}\circ H^{-1}\right)^*\left(\varphi\right),$$

which, keeping in mind that $\overline{H}\circ H^{-1}$ is a real transformation (see section 2.1.1), proves the reality of $\mathcal{H}^{-1}\left(\overline{\mathcal{H}\left(\varphi\right)}\right)$. This implies that

$$\left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_2 - \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_1 = 0.$$

Then, in view of (37), we get

$$\left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_1\right)_2 = \left(\mathcal{H}^{-1}\left(\mathcal{H}\left(\varphi\right)\right)_2\right)_1 = 0.$$

■

Now it is possible to prove the following

Theorem 6 *The linear map*

$$\Theta : \mathrm{Sym}_{\mathcal{R}}(\mathcal{Y}) \to \mathrm{Sym}_{\mathcal{R}}(\mathcal{E}) \ , \ \varphi \mapsto (\mathcal{H}(\varphi))_1 + (\mathcal{H}(\varphi))_2$$

is a vector space isomorphism.

Proof. The map $\Theta' : \eta \mapsto (\mathcal{H}^{-1}(\eta))_1 - (\mathcal{H}^{-1}(\eta))_2$ is both the left and right inverse map of Θ. In fact

$$\begin{aligned}\Theta'(\Theta(\varphi)) = &\left(\mathcal{H}^{-1}(\mathcal{H}(\varphi))_1\right)_1 + \left(\mathcal{H}^{-1}(\mathcal{H}(\varphi))_1\right)_2 + \\ &- \left(\mathcal{H}^{-1}(\mathcal{H}(\varphi))_2\right)_1 - \left(\mathcal{H}^{-1}(\mathcal{H}(\varphi))_2\right)_2\end{aligned}$$

which is equal to φ in view of previous lemma. The same reasoning holds for $\Theta(\Theta'(\eta))$. ■

Proposition 7 *Let* $\varphi, \psi \in \mathcal{R}(J^\infty(\pi))$. *Then* $\mathcal{H}\{\varphi,\psi\} = \{\mathcal{H}(\varphi), \mathcal{H}(\psi)\}$.

Proof. The statement follows from definition (8) and equation (36). ■

Remark 8 *The map* Θ *is not a Lie algebra morphism, as a direct calculation shows.*

Now we shall reproduce similar results for recursion operators. Firstly, denote by $\mathrm{Rec}_{\mathcal{R}}(\mathcal{Y})$ and $\mathrm{Rec}_{\mathcal{R}}(\mathcal{E})$, respectively, the Lie algebra of recursion operators of $\mathcal{Y}$ and $\mathcal{E}$ with coefficients in $\mathcal{R}(J^\infty(\pi))$. Similarly to the case of symmetries, we have the following

Proposition 9 *Let* $\Re \in \mathrm{Rec}_{\mathcal{R}}(\mathcal{Y})$. *Then both* $(H^{-1}(\Re))_1$ *and* $(H^{-1}(\Re))_2$ *belong to* $\mathrm{Rec}_{\mathcal{R}}(\mathcal{E})$.

Proof. It is a straightforward application of definitions and of lemma 4. ■

Theorem 10 *The linear map*

$$\Psi : \mathrm{Rec}_{\mathcal{R}}(\mathcal{Y}) \to \mathrm{Rec}_{\mathcal{R}}(\mathcal{E}) \, , \, \Re \mapsto \left(H^{-1}(\Re)\right)_1 + \left(H^{-1}(\Re)\right)_2$$

is a vector space isomorphism.

Proof. The proof is similar to the proof of theorem 6, taking into account that

$$\left(H\left(H^{-1}(\Re)\right)_i\right)_j \varphi = \left(\mathcal{H}^{-1}(\mathcal{H}(\Re(\varphi)))_i\right)_j \ , \quad i,j = 1,2.$$

■

Remark 11 *We note that all results obtained in this section remain true if one replaces in (20) matrix* P *(see (18)) with the more general matrix* $P \circ Q$ *with* Q *being any* 2×2 *real matrix.*

3 Examples of computations and Lie structure of rational symmetries of $\mathcal{E}_{ED}$

Below we apply the results obtained in the previous sections to $\mathcal{E}_{ED}$. In order to do this, in what follows by H we mean the infinite contact prolongation of $H_0 \circ G$ where we recall that H_0 is defined by (20) and G by (35). The H so defined transforms $\mathcal{E}_{ED}^{\infty}$ into $\mathcal{Y}_{ED}^{\infty}$ (see also remark 2). Also, in view of remark 11, all theorems of section 2.2 remain true for such H. We choose $(x, y, u, u_\xi, u_\eta, \ldots, u_{k\xi}, u_{k\eta}, \ldots)$ as internal coordinates on $\mathcal{Y}_{ED}^{\infty}$ and $(x, y, u, u_x, u_y, \ldots, u_{x,(k-1)y}, u_{ky}, \ldots)$ as internal coordinates on $\mathcal{E}_{ED}^{\infty}$.

As an example of computation of symmetries of $\mathcal{E}_{ED}$ we give the following

Proposition 12 *The symmetries of $\mathcal{E}_{ED}$ depending on derivatives up to second order are linearly generated by the following ones:*

$$
\begin{aligned}
X_1 &= (u_x + u_y + 2u_{xy}x + 2u_{xy}y)/(x+y)\\
X_2 &= (-yu_x + 3xu_y - 2u_{xy}y^2 + 2u_{xy}x^2 + 2u_{yy}x^2 + 4u_{yy}yx +\\
&\quad 2u_{yy}y^2 + xu_x + yu_y)/(x+y)\\
X_3 &= (-u_xyx - u_yy^2 + u_yx^2 - u_yyx - 2yx^2u_{xy} - 2y^2xu_{xy} + u_{yy}x^3 +\\
&\quad u_{yy}x^2y - u_{yy}xy^2 - u_{yy}y^3)/(x+y)\\
X_4 &= (3x^2yu_x + x^3u_x - 3y^3u_y + uy^2 - ux^2 + 9x^2yu_y - 3xy^2u_x + 8u_{xy}x^3y +\\
&\quad 4u_{yy}x^3y + 4u_{yy}y^3x - 8u_{xy}y^3x + 12u_{yy}x^2y^2 + 2u_{xy}x^4 - 2u_{xy}y^4 -\\
&\quad 2u_{yy}x^4 - 2u_{yy}y^4 + 3xy^2u_y - x^3u_y - y^3u_x)/(x+y)\\
X_5 &= (uy^3 + ux^3 - 12u_{xy}x^2y^3 - 12u_{xy}x^3y^2 - 20u_yy^3x + 12u_yx^3y -\\
&\quad 18u_yx^2y^2 + 4u_xy^3x - 12u_xx^3y - 18u_xx^2y^2 - 5uxy^2 - 5ux^2y +\\
&\quad u_yy^4 + 5u_yx^4 + u_xx^4 + 5u_xy^4 - 8u_{yy}y^4x - 8u_{yy}y^3x^2 + 8u_{yy}y^2x^3 +\\
&\quad 8u_{yy}yx^4 + 2u_{xy}y^4x + 2u_{xy}x^4y + 2u_{xy}x^5 + 2u_{xy}y^5)/(x+y)\\
X_6 &= -u_x + u_y\\
X_7 &= u + 2xu_x + 2yu_y\\
X_8 &= ux - uy - u_xy^2 + u_xx^2 - 2u_xyx + u_yx^2 - u_yy^2 + 2u_yyx\\
X_9 &= u
\end{aligned}
$$

Proof. Firstly we take symmetries depending on second derivatives of $\mathcal{Y}_{ED}$, obtained in [11], and then we transform them by using the map Θ (see theorem 6). ■

Symmetries X_6, X_7, X_8 and X_9 generate the algebra of classical symmetries of $\mathcal{E}_{ED}$. We note that they are point symmetries, as their generating sections are linear in the first derivatives. This means that the corresponding vector fields on $J^2(\pi)$ are prolongations of vector fields on E rather than on $J^1(\pi)$. This fact was noticed in [2], where variational aspects of $\mathcal{E}_{ED}$ were also studied.

Let us go back to equation $\mathcal{Y}_{ED}$. In [10],[11],[12] it is proved that contact symmetries coincide with classical point symmetries. More precisely we have that the most general

generating section of a contact symmetry is, up to an arbitrary solution, of the following form:

$$\varphi = \left(c_1 \frac{(\eta - \xi)}{2} + c_4 \right) u + \left(-c_1 \xi^2 + c_2 \xi - c_3 \right) u_\xi + \left(c_1 \eta^2 + c_2 \eta + c_3 \right) u_\eta$$

where c_1, c_2, c_3, c_4 are arbitrary constants. As equation $\mathcal{Y}_{ED}$ is linear, such symmetries determine recursion operators as they are linear in u and in its derivatives ([8]). In particular

$$\Box = D_\xi - D_\eta, \quad \sigma = \xi D_\xi + \eta D_\eta + \frac{\mathbb{I}}{2}, \quad \tau = \xi^2 D_\xi - \eta^2 D_\eta + \frac{\xi - \eta}{2} \mathbb{I} \tag{39}$$

are recursion operators. Since higher symmetries of $\mathcal{Y}_{ED}$ are linear in u and in its derivatives (see theorem 13 below), then in this case the theory of higher symmetries can be developed using recursion operators as fundamental objects ([8]). For instance, it is natural to ask if by applying arbitrary compositions of (39) to the symmetry u we get the whole algebra of higher symmetries. For this purpose, let us consider

$$\Box_j^m = [\dots [\Box^m, \underbrace{\tau], \dots, \tau]}_{\text{j-times}} \tag{40}$$

and

$$\varphi_0 = \Box(u), \quad \varphi_1 = \tau(u), \quad \varphi_2 = \xi u_{\xi\xi} - \eta u_{\eta\eta} + \frac{\xi u_\xi - \eta u_\eta}{\xi + \eta}.$$

Since the recursion operators (39) are $\mathcal{C}$-differential operators, it is well defined the restriction $\overline{\Box}_j^m$ of (40) on the equation. The following three theorems are due to Shemarulin (see [10],[11],[12]).

Theorem 13 *Let $\varphi \in C^\infty \left(\mathcal{Y}_{ED}^{n-2} \right)$ be a symmetry of $\mathcal{Y}_{ED}$. Then φ has the following form:*

$$\varphi = \phi(\xi, \eta) + \sum_{0 \le k \le n} \mathcal{P}_k(\xi, \eta) u_{k\xi} + \sum_{0 \le h \le n} \mathcal{Q}_h(\xi, \eta) u_{h\eta}, \tag{41}$$

where ϕ is a solution of $\mathcal{Y}_{ED}$ and $\mathcal{P}_k$ and $\mathcal{Q}_h$ are rational functions.

Theorem 14 *The algebra* $\mathrm{Sym}\,(\mathcal{Y}_{ED})$ *is the semi-direct sum* $A \oplus \mathrm{NSym}\,(\mathcal{Y}_{ED})$ *where A is the abelian infinite dimensional ideal of solutions of $\mathcal{Y}_{ED}$ and* $\mathrm{NSym}\,(\mathcal{Y}_{ED})$ *is the algebra linearly generated by u and φ_j^m where*

$$\varphi_j^m = \{ \dots \{ \{ \dots \{ \varphi_0, \underbrace{\varphi_2 \} \dots \varphi_2}_{\text{(j-1)-times}} \} \underbrace{\varphi_1 \} \dots \varphi_1}_{\text{m-times}} \}.$$

Theorem 15 *The algebra* $\mathrm{NSym}\,(\mathcal{Y}_{ED})$ *is linearly generated by u and $\overline{\Box}_j^m(u)$. Moreover, we have the following relations:*

$$\begin{aligned}
\left[\Box_j^m, \Box \right] &= -j\,(2m - j + 1)\, \Box_{j-1}^m, & 1 \le j \le 2m; \\
\left[\Box_j^m, \sigma \right] &= (m - j)\, \Box_j^m, & 0 \le j \le 2m; \\
\left[\Box_j^m, \tau \right] &= \Box_{j+1}^m, & 0 \le j \le 2m;
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\Box}_j^m(u) \neq 0, && 0 \leq j \leq 2m; \\
&\overline{\Box}_j^m(u) = 0, && j \geq 2m+1.
\end{aligned}$$

Now we shall reproduce, for the equation $\mathcal{E}_{ED}$, similar results by using the complex transformation H.

Proposition 16 *Rational symmetries of $\mathcal{E}_{ED}$ are linear in the internal jet variables.*

Proof. We recall (theorem 6) that rational symmetries of $\mathcal{E}_{ED}$ are images through $\mathcal{H}$ of symmetries of $\mathcal{Y}_{ED}$, which have the form (41). In view of the fact H is a block matrix, the proposition follows if we show that the restriction of $u_{mx,ny}$ to $\mathcal{E}_{ED}^{m+n-2}$ is a linear function in the internal jet variables. We show it by induction. Firstly, a straightforward computation shows that

$$u_{xx,hy}\big|_{\mathcal{E}_{ED}^h} = -\sum_{k=0}^{h} \binom{h}{k} \frac{(-1)^k k!}{(x+y)^{k+1}} \left(u_{x,(h-k)y} + u_{(h-k+1)y}\right) - u_{(h+2)y}. \tag{42}$$

Now let us suppose that $u_{(m-1)x,ny}\big|_{\mathcal{E}_{ED}^{m+n-3}}$ is linear in the internal jet variables. Namely

$$u_{(m-1)x,ny}\Big|_{\mathcal{E}_{ED}^{m+n-3}} = \sum_{h=0}^{m+n-2} a_h(x,y) u_{x,hy} + \sum_{j=0}^{m+n-1} b_j(x,y) u_{jy}.$$

Then

$$\begin{aligned}
u_{mx,ny}\big|_{\mathcal{E}_{ED}^{m+n-2}} &= \bar{D}_x\left(u_{(m-1)x,ny}\big|_{\mathcal{E}_{ED}^{m+n-1}}\right) \\
&= \sum_{h=0}^{m+n-2}\left(\frac{\partial a_h}{\partial x} u_{x,hy} + a_h\, u_{xx,hy}\big|_{\mathcal{E}_{ED}^h}\right) + \sum_{j=0}^{m+n-1}\left(\frac{\partial b_j}{\partial x} u_{jy} + b_j u_{x,jy}\right).
\end{aligned}$$

The assertion is proved in view of (42). ■

We define the algebra $\mathrm{NSym}\,(\mathcal{E}_{ED})$ as the algebra of rational higher symmetries of $\mathcal{E}_{ED}$ up to solutions of $\mathcal{E}_{ED}$. As we have noticed in remark 8, the map Θ is not a Lie algebra morphism. Anyway we have the following

Proposition 17 *The algebra* $\mathrm{NSym}\,(\mathcal{E}_{ED})$ *is infinite dimensional as vector space and finitely generated as Lie algebra. More precisely*

$$\varrho_j^m = \{\dots\{\{\dots\{\varrho_0, \underbrace{\varrho_2\}\dots\varrho_2\}}_{\text{(j-1)-times}}\underbrace{\varrho_1\}\dots\varrho_1\}}_{\text{m-times}}$$

where

$$\begin{aligned}
\varrho_0 &= -u_x + u_y \\
\varrho_1 &= \frac{1}{2}\left(x^2 - 2xy - y^2\right) u_x + \frac{1}{2}\left(x^2 + 2xy - y^2\right) u_y + \frac{1}{2}(x-y)\, u \\
\varrho_2 &= (x+y)\, u_{yy} + (x-y)\, u_{xy} + \frac{1}{2(x+y)}\left((x-y)\, u_x + (3x+y)\, u_y\right)
\end{aligned}$$

form, together with u, a linear basis of $\mathrm{NSym}\,(\mathcal{E}_{ED})$.

Proof. The proposition follows taking into consideration that Θ restricts on NSym $(\mathcal{Y}_{ED})$ and NSym $(\mathcal{E}_{ED})$, theorem 14, and finally that $\mathcal{H}(\varphi_0) = i\varrho_0$, $\mathcal{H}(\varphi_1) = i\varrho_1$, $\mathcal{H}(\varphi_2) = i\varrho_2$. ■

By transforming recursion operators (39) throughout H^{-1}, we get the following

Proposition 18

$$H^{-1}(\Box) = i\widetilde{\Box}, \quad H^{-1}(\tau) = i\widetilde{\tau}, \quad H^{-1}(\sigma) = \widetilde{\sigma},$$

where

$$\begin{aligned}
\widetilde{\Box} &= -D_x + D_y \\
\widetilde{\tau} &= \left(\frac{1}{2}x^2 - xy - \frac{1}{2}y^2\right) D_x + \left(\frac{1}{2}x^2 + xy - \frac{1}{2}y^2\right) D_y + \frac{1}{2}(x - y)\,\mathbb{I} \\
\widetilde{\sigma} &= xD_x + yD_y + \frac{\mathbb{I}}{2}
\end{aligned}$$

Now, if we define

$$\triangledown_j^m = [\dots[\widetilde{\Box}^m, \underbrace{\widetilde{\tau}], \dots, \widetilde{\tau}]}_{\text{j-times}}$$

we get the Lie structure of NSym $(\mathcal{E}_{ED})$ by means of the following

Theorem 19 *The algebra* NSym $(\mathcal{E}_{ED})$ *is linearly generated by* u *and* $\overline{\nabla}_j^m(u)$*. Moreover, we have the following relations:*

$$\begin{aligned}
\left[\nabla_j^m, \widetilde{\Box}\right] &= j\,(2m - j + 1)\,\nabla_{j-1}^m, && 1 \le j \le 2m; \\
\left[\nabla_j^m, \widetilde{\sigma}\right] &= (m - j)\,\nabla_j^m, && 0 \le j \le 2m; \\
\left[\nabla_j^m, \widetilde{\tau}\right] &= \nabla_{j+1}^m, && 0 \le j \le 2m;
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}_j^m(u) &\neq 0, && 0 \le j \le 2m; \\
\overline{\nabla}_j^m(u) &= 0, && j \ge 2m + 1.
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
H^{-1}\left(\Box_j^m\right) &= [\dots[\left(H^{-1}(\Box)\right)^m, \underbrace{H^{-1}(\tau)], \dots, H^{-1}(\tau)]}_{\text{j-times}} \\
&= i^{m+j}[\dots[\widetilde{\Box}^m, \underbrace{\widetilde{\tau}], \dots, \widetilde{\tau}]}_{\text{j-times}} = i^{m+j}\,\triangledown_j^m\,.
\end{aligned}$$

Then, taking into account theorem 10 and theorem 15, the theorem follows. ■

Finally, taking into account that symmetries of $\mathcal{E}_{ED}$ are linear in u_σ (see proposition 16), for each couple Δ, ∇ of recursion operators of $\mathcal{E}_{ED}$, we have that $[\Delta, \nabla](u) = -\{\Delta(u), \nabla(u)\}$. Then, in view of previous theorem, we get the Lie structure of NSym $(\mathcal{E}_{ED})$.

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